GL(3)-Based Quantum Integrable Composite Models. I. Bethe Vectors

Stanislav PAKULIAK [abc], *Eric RAGOUCY* [d] *and Nikita A. SLAVNOV* [e]

[a] *Institute of Theoretical & Experimental Physics, 117259 Moscow, Russia*

[b] *Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow reg., Russia*
E-mail: *pakuliak@jinr.ru*

[c] *Moscow Institute of Physics and Technology, 141700, Dolgoprudny, Moscow reg., Russia*

[d] *Laboratoire de Physique Théorique LAPTH, CNRS and Université de Savoie, BP 110, 74941 Annecy-le-Vieux Cedex, France*
E-mail: *eric.ragoucy@lapth.cnrs.fr*

[e] *Steklov Mathematical Institute, Moscow, Russia*
E-mail: *nslavnov@mi.ras.ru*

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Abstract. We consider a composite generalized quantum integrable model solvable by the nested algebraic Bethe ansatz. Using explicit formulas of the action of the monodromy matrix elements onto Bethe vectors in the GL(3)-based quantum integrable models we prove a formula for the Bethe vectors of composite model. We show that this representation is a particular case of general coproduct property of the weight functions (Bethe vectors) found in the theory of the deformed Knizhnik–Zamolodchikov equation.

Key words: Bethe ansatz; quantum affine algebras, composite models

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1 Introduction

Solution of the inverse scattering problem for XXZ Heisenberg spin-1/2 chain in the pioneering paper [9] opened a possibility to apply the results of calculations of the Bethe vectors scalar products [14] to the problem of calculation of the correlation functions for some class of quantum integrable models (see review [15] and references therein). This class of integrable models is characterized by the property that the monodromy operator is constructed from the R-matrix and it becomes a permutation operator at some value of the spectral parameter.

For quantum models where such property is absent there exists another method to calculate the correlation functions of the local operators [4]. The main idea of this method is to divide artificially the interval where model is defined into two sub-intervals. In this framework one has a possibility to study form factors of operators depending on an internal point of the original interval. This approach was used to analyze some correlation functions in the bose gas model that describes one-dimensional bosons with δ-function interaction [5, 8]. The authors of [4] called such the model *two-site* generalized model. Later, in [15] the terminology *two-component* model was used. We think that both terms might be misleading: the first one if we apply this approach to the study of lattice models (like spin chains, where 'site' has already a definite and different meaning); the second if we use such model for the description of multi-component gases, where 'component' refers to the different types of particles. Therefore in this paper we call this model a *composite* model.

In the composite model the total monodromy matrix $T(u)$ is presented as a usual matrix product of the partial monodromy matrices $T^{(2)}(u)$ and $T^{(1)}(u)$:

$$T(u) = T^{(2)}(u)T^{(1)}(u). \tag{1.1}$$

The matrix elements of $T(u)$ are operators in the space of states V that corresponds to an interval $[0, L]$. The matrix elements of the partial monodromy matrices $T^{(1)}(u)$ and $T^{(2)}(u)$ act in the spaces $V^{(1)}$ and $V^{(2)}$ corresponding to the intervals $[0, x]$ and $]x, L]$ respectively. Here x is an intermediate point of the interval $[0, L]$. The total space of states V is a tensor product of the partial spaces of states $V^{(1)} \otimes V^{(2)}$. We assume that each of these spaces possesses unique vacuum vector $|0\rangle^{(\ell)}$ defined by the formulas

$$\begin{aligned} &T_{ij}(u)|0\rangle = 0, \quad i > j, &&T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle,\\ &T^{(\ell)}_{ij}(u)|0\rangle^{(\ell)} = 0, \quad i > j, &&T^{(\ell)}_{ii}(u)|0\rangle^{(\ell)} = \lambda^{(\ell)}_i(u)|0\rangle^{(\ell)}, \quad \ell = 1, 2, \end{aligned} \tag{1.2}$$

and $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$.

We also assume that dual spaces $V^{*(1)}$ and $V^{*(2)}$ possess a dual vacuum vectors $\langle 0|^{(\ell)}$ with analogous properties

$$\begin{aligned} &\langle 0|T_{ij}(u) = 0, \quad i < j, &&\langle 0|T_{ii}(u) = \lambda_i(u)\langle 0|,\\ &\langle 0|^{(\ell)}T^{(\ell)}_{ij}(u) = 0, \quad i < j, &&\langle 0|^{(\ell)}T^{(\ell)}_{ii}(u) = \lambda^{(\ell)}_i(u)\langle 0|^{(\ell)}, \quad \ell = 1, 2, \end{aligned}$$

and $\langle 0| = \langle 0|^{(1)} \otimes \langle 0|^{(2)}$.

Spaces of states $V^{(1)}$, $V^{(2)}$ in each part of the composite model are formed by partial Bethe vectors $\mathbb{B}^{(\ell)} \in V^{(\ell)}$, $\ell = 1, 2$. They are given by certain polynomials in the entries of the partial monodromy matrices $T^{(\ell)}_{ij}(u)$, $i < j$ acting onto vacuum vector $|0\rangle^{(\ell)}$. Total Bethe vectors $\mathbb{B} \in V$ are the same polynomials in the entries of the total monodromy matrix acting onto $|0\rangle$. Bethe vectors (total and partial) depend on complex variables called Bethe parameters (see, e.g., (2.7)). Total Bethe vectors are characterized by the property that at certain values of Bethe parameters they become eigenvectors of a total transfer matrix $\operatorname{tr} T(u)$, where the trace is taken in the matrix space. Similarly, partial Bethe vectors are eigenvectors of partial transfer matrices $\operatorname{tr} T^{(\ell)}(u)$ at certain values of their Bethe parameters.

The main advantage of the composite model is that it allows one to calculate form factors of the partial monodromy matrix elements $T^{(\ell)}_{ij}(u)$ in the basis of the total Bethe vectors. This opens a way for computing form factors and correlation functions of local operators [4, 5, 8]. However, for this purpose, one should express the total Bethe vectors in terms of the partial ones. Such representation is a necessary tool for all further studies.

This problem was solved for GL(2)-based models in [4]. There it was shown that the total Bethe vector is a linear combination of the partial Bethe vectors tensor products. This is not a surprising result, because in the GL(2)-based models Bethe vectors have a very simple structure: they are monomials in the operator $T_{12}(u)$ acting on $|0\rangle$. Since due to (1.1) we have

$$T_{ij}(u) = \big(T^{(2)}(u) \cdot T^{(1)}(u)\big)_{ij} = \sum_{k=1}^{2} T^{(2)}_{ik}(u)T^{(1)}_{kj}(u), \tag{1.3}$$

we find, in particular, $T_{12}(u) = T^{(2)}_{11}(u)T^{(1)}_{12}(u) + T^{(2)}_{12}(u)T^{(1)}_{22}(u)$. It is clear therefore that the action of any monomial in $T_{12}(u)$ on the vacuum reduces to a bilinear combination of monomials in $T^{(\ell)}_{12}(u)$ acting on partial vacuums[1]. One should only find the coefficients of this bilinear combination. It was done in [4] (see (2.14) below).

[1]Note that in (1.3) the operators $T^{(2)}_{ik}(u)$ and $T^{(1)}_{kj}(v)$ commute with each other, as they act in different spaces.

The analogous problem for GL(N)-based models with $N > 2$ is more sophisticated. In these models Bethe vectors have much more complex structure (see, e.g., (2.7) for $N = 3$). Therefore, the possibility to express total Bethe vectors in terms of partial ones does not look obviously solvable.

This problem was studied and solved in the theory of the deformed Knizhnik–Zamolodchikov (KZ) equations [16]. In this theory one of the building blocks to construct the integral solutions to these equations is a so-called *weight function*. The weight function appears to be nothing but a Bethe vector for the model whose monodromy matrix is constructed from the quantum R-matrices corresponding to the deformed KZ equation. The weight function is defined in the finite-dimensional representation space of the corresponding Yang–Baxter algebra of the monodromy operators $T_{ij}(u)$. It should satisfy a coproduct property. This property states that if the weight function is known for two representation spaces $V^{(1)}$ and $V^{(2)}$, then it can be uniquely determined for the representation space $V^{(1)}\otimes V^{(2)}$ (see (5.3)). If monodromy operators are composed from the generators of some finite [16] or infinite [6] algebras with Hopf structures, the coproduct property of the weight functions is a direct consequence of the structure of the Bethe vectors and the coproduct property of the monodromy operators

$$\Delta T_{ij}(u) = \sum_{k=1}^{3} T_{kj}(u) \otimes T_{ik}(u), \tag{1.4}$$

applied to the tensor product $V^{(1)}\otimes V^{(2)}$. According to the algebraic approach [6, 16] monodromy operator for the composite model is defined by the right hand side of the coproduct formula (1.4) and using Sweedler notations this can be rewritten

$$\sum_{k=1}^{3} T^{(2)}_{ik}(u) T^{(1)}_{kj}(u) = \big(T^{(2)}(u)\cdot T^{(1)}(u)\big)_{ij} \tag{1.5}$$

in the form of the matrix multiplication of the partial monodromies of the composite model. Comparing (1.5) with (1.1) we see that the usual matrix product in the auxiliary space can be interpreted as the coproduct in a quantum space, so that (1.1) can be recast as $\Delta T(u) = T^{(2)}(u)T^{(1)}(u)$. Thus, one can consider the relationship between total and partial Bethe vectors as a coproduct property of the Bethe vectors.

In [16] this property was proved by use of the trace formula for the Bethe vectors and coproduct formulas (1.4). Another approach to this problem was developed in [2]. There a *universal* weight function was identified with certain projections of the product of the $U_q(\widehat{\mathfrak{gl}}_N)$ currents onto intersections of the Borel subalgebras of different type in the quantum affine algebras. It is worth mentioning that this method also allows one to obtain explicit formulas for the Bethe vectors in GL(N)-based quantum integrable models [7, 6, 11] in terms of the polynomials of the matrix elements of the monodromy applied to the vacuum vectors.

In this paper we develop one more approach of finding an expression of total Bethe vectors in terms of partial Bethe vectors in GL(3)-invariant composite model. We show that this expression can be found directly from the algebra of the monodromy operators (2.2). More precisely, we use the formulas of the action of the monodromy matrix elements onto the Bethe vectors found in [1]. As a result, we present the total Bethe vectors $\mathbb{B}$ as a linear combination of the tensor products of the partial Bethe vectors[2] $\mathbb{B}^{(\ell)}$. The use of morphisms of the algebra (2.2) also allows us to obtain explicit expressions for dual Bethe vectors of the composite model. The last ones are necessary for calculating form factors of the partial monodromy matrix entries $T^{(\ell)}_{ij}(u)$.

[2] In what follows we do not write the symbol $\otimes$ in the tensor product of the partial Bethe vectors $\mathbb{B}^{(1)}\mathbb{B}^{(2)}$. Instead we distinguish the tensor components by the superscripts $^{(1)}$ and $^{(2)}$.

Finally, we show that our results agree with the coproduct properties of the universal weight functions (5.3).

The paper is organized as follows. In Section 2 we formulate the main statement and reduce the proof of this assertion to the calculation of the action of the monodromy matrix elements $T_{13}(z)$ and $T_{12}(z)$ onto certain combination of the partial Bethe vectors $\mathbb{B}^{(\ell)}$. These actions are considered in Sections 3 and 4. Then in Section 5 we specialize the general coproduct properties of the universal weight function in $\mathrm{GL}(N)$-based model to the $\mathrm{GL}(3)$ case. It demonstrates that the direct calculations by the action formulas and coproduct properties of the universal weight function are well correlated. Appendix A gathers formulas for the action of the monodromy matrix elements onto Bethe vectors. Appendix B collects necessary calculations for the statements of Section 4.

2 Coproduct property of Bethe vectors

The quantum integrable models considered in this paper are related to the quantum $\mathrm{GL}(3)$-invariant R-matrix

$$R(x,y) = \mathbf{1} + g(x,y)\mathbb{P}, \qquad g(x,y) = \frac{c}{x-y}, \tag{2.1}$$

where $\mathbb{P}$ is a permutation operator of two three-dimensional spaces $\mathbb{C}^3$ and c is a constant. The total monodromy matrix $T(u)$ satisfies the RTT algebra

$$R(w_1,w_2)\cdot(T(w_1)\otimes 1)\cdot(1\otimes T(w_2)) = (1\otimes T(w_2))\cdot(T(w_1)\otimes 1)\cdot R(w_1,w_2). \tag{2.2}$$

Every partial monodromy matrix $T^{(\ell)}(u)$ also satisfies the RTT-relation (2.2) with R-matrix (2.1). Matrix elements of the monodromy are introduced by the sum $T(u) = \sum_{i,j=1}^{3} \mathsf{E}_{ij} T_{ij}(u)$, where E_{ij} are the matrix units with 1 at the intersection of i^{th} row and j^{th} column and zero elsewhere. In terms of the matrix units R-matrix (2.1) has the form

$$R(x,y) = \sum_{i,j=1}^{3} \mathsf{E}_{ii}\otimes\mathsf{E}_{jj} + g(x,y)\sum_{i,j=1}^{3}\mathsf{E}_{ij}\otimes\mathsf{E}_{ji}.$$

Instead of the functions $\lambda_i(u)$, $\lambda_i^{(1)}(u)$ and $\lambda_i^{(2)}(u)$ defined by (1.2) we will use their ratios

$$r_k(u) = \frac{\lambda_k(u)}{\lambda_2(u)}, \qquad r_k^{(\ell)}(u) = \frac{\lambda_k^{(\ell)}(u)}{\lambda_2^{(\ell)}(u)}, \qquad \ell = 1,2, \quad k = 1,3.$$

It is obvious from equation (1.1) and definition (1.2) of the vacuum vectors that

$$\lambda_i(u) = \lambda_i^{(1)}(u)\lambda_i^{(2)}(u), \qquad r_k(u) = r_k^{(1)}(u) r_k^{(2)}(u). \tag{2.3}$$

In this paper we use the same notation and conventions as in [1]. Besides the rational function $g(x,y)$ we also use another rational function

$$f(x,y) = 1 + g(x,y) = \frac{x-y+c}{x-y}. \tag{2.4}$$

We denote sets of variables by bar: $\bar u$, $\bar v$ and so on. We also consider partitions of sets into disjoint subsets and denote them by symbol $\Rightarrow$. For example, the notation $\bar u \Rightarrow \{\bar u_{\rm I}, \bar u_{\rm II}\}$

means that the set $\bar u$ is divided into two disjoint subsets[3] $\bar u_{\rm I}$ and $\bar u_{\rm II}$, such that $\bar u_{\rm I}\cap\bar u_{\rm II}=\varnothing$ and $\{\bar u_{\rm I},\bar u_{\rm II}\}=\bar u$. Special notation $\bar u_0$, $\bar v_k$ and so on is used for the sets $\bar u\setminus u_0$, $\bar v\setminus v_k$ etc.

In order to avoid excessively cumbersome formulas we use shorthand notation for products of commuting operators $T_{ij}(u)$ or the vacuum eigenvalues $\lambda_i(u)$ or $r_k(u)$ (partial and total). Namely, whenever such an operator (or a function) depends on a set of variables, this means that we deal with the product of these operators (functions) with respect to the corresponding set, as follows:

$$T_{23}(\bar v_\ell)=\prod_{\substack{v_k\in\bar v\\ v_k\ne v_\ell}}T_{23}(v_k),\qquad \lambda_i(\bar u)=\prod_{u_j\in\bar u}\lambda_i(u_j),\qquad r_1^{(2)}(\bar u_{\rm II})=\prod_{u_j\in\bar u_{\rm II}}r_1^{(2)}(u_j). \tag{2.5}$$

This notation is also used for products of the functions $f(x,y)$ and $g(x,y)$,

$$g(w,\bar v)=\prod_{v_j\in\bar v}g(w,v_j),\qquad f(\bar u_{\rm II},\bar u_{\rm I})=\prod_{u_j\in\bar u_{\rm II}}\prod_{u_k\in\bar u_{\rm I}}f(u_j,u_k). \tag{2.6}$$

2.1 Bethe vectors

Bethe vectors of GL(3)-invariant models depend on two sets of Bethe parameters: $\mathbb{B}=\mathbb{B}_{a,b}(\bar u;\bar v)$, where $\bar u=\{u_1,\dots,u_a\}$, $\bar v=\{v_1,\dots,v_b\}$, $a,b=0,1,\dots$. Several explicit representations for Bethe vectors were found in [1]. We present here one of them using the shorthand notation introduced in (2.5), (2.6)

$$\mathbb{B}_{a,b}(\bar u;\bar v)=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}}\frac{\mathsf{K}_n(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar v_{\rm II})\lambda_2(\bar u)}\frac{f(\bar v_{\rm II},\bar v_{\rm I})f(\bar u_{\rm I},\bar u_{\rm II})}{f(\bar v,\bar u)}T_{13}(\bar u_{\rm I})T_{12}(\bar u_{\rm II})T_{23}(\bar v_{\rm II})|0\rangle. \tag{2.7}$$

Here the sum is taken over partitions of the sets $\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}$ and $\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}$. The partitions are independent except the condition $\#\bar u_{\rm I}=\#\bar v_{\rm I}=n$, where $n=0,1,\cdots\min(a,b)$. The function $\mathsf{K}_n(\bar v_{\rm I}|\bar u_{\rm I})$ is a partition function of the six-vertex model with domain wall boundary conditions [10]. It has the following explicit representation [3]:

$$\mathsf{K}_n(\bar v|\bar u)=\prod_{\ell<m}^{n}g(v_\ell,v_m)g(u_m,u_\ell)\cdot\frac{f(\bar v,\bar u)}{g(\bar v,\bar u)}\det\left[\frac{g^2(v_i,u_j)}{f(v_i,u_j)}\right]\Bigg|_{i,j=1,\dots,n}.$$

Partial Bethe vectors $\mathbb{B}^{(\ell)}_{a,b}(\bar u;\bar v)$ are given by the same formula (2.7), where one should replace all T_{ij} by $T^{(\ell)}_{ij}$, all λ_2 by $\lambda_2^{(\ell)}$, and $|0\rangle$ by $|0\rangle^{(\ell)}$, $\ell=1,2$. In order to express the total Bethe vector in terms of matrix elements $T^{(\ell)}_{ij}$ acting on $|0\rangle^{(1)}\otimes|0\rangle^{(2)}$ one should substitute (1.1) for every T_{ij} and (2.3) for all λ_2 into equation (2.7). It is a highly nontrivial fact that the result of this substitution can be reduced to a bilinear combination of partial Bethe vectors.

We denote dual Bethe vectors by $\mathbb{C}_{a,b}(\bar u;\bar v)$. They are given by a formula similar to (2.7)

$$\mathbb{C}_{a,b}(\bar u;\bar v)=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}}\frac{\mathsf{K}_n(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar v_{\rm II})\lambda_2(\bar u)}\frac{f(\bar v_{\rm II},\bar v_{\rm I})f(\bar u_{\rm I},\bar u_{\rm II})}{f(\bar v,\bar u)}\langle 0|T_{32}(\bar v_{\rm II})T_{21}(\bar u_{\rm II})T_{31}(\bar u_{\rm I}).$$

Here the notation is the same as in (2.7). In order to obtain partial dual Bethe vectors one should make the replacements already mentioned, namely $T_{ij}\to T^{(\ell)}_{ij}$, $\lambda_2\to\lambda_2^{(\ell)}$, and $\langle 0|\to\langle 0|^{(\ell)}$.

[3]To be rigorous, since $\bar u_{\rm I}$ and $\bar u_{\rm II}$ are sets, we should note $\bar u_{\rm I}\cup\bar u_{\rm II}$, but to lighten presentation when there are many subsets, we use the notation $\{\bar u_{\rm I},\bar u_{\rm II}\}$.

2.2 Morphisms of the algebra and relations between Bethe vectors

The R-matrix (2.1) is invariant under simultaneous transpositions in both auxiliary spaces. This fact implies the existence of two symmetries in the algebra of the monodromy matrix elements. The mapping

$$\psi:\ T_{ij}(u)\to T_{ji}(u) \tag{2.8}$$

defines an antimorphism of the algebra (2.2), while the mapping

$$\varphi:\ T_{ij}(u)\to T_{4-j,4-i}(-u) \tag{2.9}$$

defines an isomorphism of the algebra (2.2).

Assuming that

$$\psi(|0\rangle)=\langle 0|,\qquad \psi(\langle 0|)=|0\rangle,\qquad \varphi(|0\rangle)=|0\rangle,\qquad \varphi(\langle 0|)=\langle 0|$$

we obtain from the mappings (2.8) and (2.9) the following relations between different Bethe vectors [1]

$$\psi(\mathbb{B}_{a,b}(\bar u;\bar v))=\mathbb{C}_{a,b}(\bar u;\bar v),\qquad \psi(\mathbb{C}_{a,b}(\bar u;\bar v))=\mathbb{B}_{a,b}(\bar u;\bar v), \tag{2.10}$$

and

$$\varphi(\mathbb{B}_{a,b}(\bar u;\bar v))=\mathbb{B}_{b,a}(-\bar v;-\bar u),\qquad \varphi(\mathbb{C}_{a,b}(\bar u;\bar v))=\mathbb{C}_{b,a}(-\bar v;-\bar u).$$

Moreover the mappings (2.8) and (2.9) intertwine the coproduct (1.4) and inverse coproduct

$$\Delta' T_{ij}(u)=\sum_{k=1}^{3}T_{ik}(u)\otimes T_{kj}(u).$$

Namely

$$\Delta\circ\psi=(\psi\otimes\psi)\circ\Delta',\qquad \Delta\circ\varphi=(\varphi\otimes\varphi)\circ\Delta'. \tag{2.11}$$

For example, the first intertwining relation follows from the chain of equalities

$$\begin{aligned}\Delta\circ\psi(T_{ij}(u))=\Delta(T_{ji}(u))&=\sum_{k=1}^{3}T_{ki}(u)\otimes T_{jk}(u)\\ &=\sum_{k=1}^{3}\psi(T_{ik}(u))\otimes\psi(T_{kj}(u))=(\psi\otimes\psi)\circ\Delta'(T_{ij}(u)).\end{aligned}$$

The second property in (2.11) can be proved in the same way. Thus, we see that the action of the mappings ψ and φ exchanges the components in the tensor product.

2.3 Main statement

Theorem 2.1. *The Bethe vectors of the total monodromy matrix $T(u)$ can be presented as a bilinear combination of partial Bethe vectors as follows:*

$$\mathbb{B}_{a,b}(\bar u;\bar v)=\sum r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}\mathbb{B}^{(1)}_{a_1,b_1}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}_{a_2,b_2}(\bar u_{\rm II};\bar v_{\rm II}). \tag{2.12}$$

Here the sums are taken over all the partitions $\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}$ and $\bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}$. The integers a_ℓ and b_ℓ ($\ell=1,2$) are the cardinalities of the corresponding subsets. Hereby, $a_1+a_2=a$ and $b_1+b_2=b$.

Sketch of proof. We use an induction over a. First, for arbitrary $a, b \geq 0$ we define a vector $\mathcal{B}_{a,b}(\bar u, \bar v)$ as follows:

$$\mathcal{B}_{a,b}(\bar u; \bar v) = \sum r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})} \mathbb{B}^{(1)}_{a_1,b_1}(\bar u_{\rm I}; \bar v_{\rm I}) \mathbb{B}^{(2)}_{a_2,b_2}(\bar u_{\rm II}; \bar v_{\rm II}). \tag{2.13}$$

When $a = 0$ and b is arbitrary, we obtain

$$\mathcal{B}_{0,b}(\varnothing; \bar v) = \sum r_3^{(1)}(\bar v_{\rm II}) f(\bar v_{\rm II}, \bar v_{\rm I}) \mathbb{B}^{(1)}_{0,b_1}(\varnothing; \bar v_{\rm I}) \mathbb{B}^{(2)}_{0,b_2}(\varnothing; \bar v_{\rm II}). \tag{2.14}$$

This is a known formula for the Bethe vector in the composite model in the GL(2) case [4]. Thus, we conclude that $\mathcal{B}_{0,b}(\varnothing; \bar v) = \mathbb{B}_{0,b}(\varnothing; \bar v)$.

Then we consider the action of the operators $T_{13}(z)$ and $T_{12}(z)$ onto this vector. The goal is to prove that

$$\frac{T_{13}(z)}{\lambda_2(z)} \mathcal{B}_{a-1,b-1}(\bar u; \bar v) = \mathcal{B}_{a,b}(\{\bar u, z\}; \{\bar v, z\}), \tag{2.15}$$

and

$$\frac{T_{12}(z)}{\lambda_2(z)} \mathcal{B}_{a-1,b}(\bar u; \bar v) = f(\bar v, z) \mathcal{B}_{a,b}(\{\bar u, z\}; \bar v) + \sum g(z, v_0) f(\bar v_0, v_0) \mathcal{B}_{a,b}(\{\bar u, z\}; \{\bar v_0, z\}), \tag{2.16}$$

where the sum is taken over partitions $\bar v \Rightarrow \{v_0, \bar v_0\}$ with $\#v_0 = 1$ (recall that due to our convention $\bar v_0 = \bar v \setminus v_0$).

If these actions are proved, then we obtain a recursion

$$\mathcal{B}_{a,b}(\{\bar u, z\}; \bar v) = \frac{T_{12}(z) \mathcal{B}_{a-1,b}(\bar u; \bar v)}{\lambda_2(z) f(\bar v, z)} + \sum g(z, v_0) f(\bar v_0, v_0) \frac{T_{13}(z) \mathcal{B}_{a-1,b-1}(\bar u; \bar v_0)}{\lambda_2(z) f(\bar v, z)}. \tag{2.17}$$

Since this recursion coincides with the one for the Bethe vectors (see, e.g., [1]), and using the equality proven above for $a = 0$, we conclude that $\mathcal{B}_{a,b}$ is a Bethe vector $\mathbb{B}_{a,b}$. Thus, the proof of Theorem 2.1 reduces to proving equations (2.15) and (2.16): it is done in Sections 3 and 4. ■

Corollary 2.2. *Dual Bethe vectors of the total monodromy matrix $T(u)$ can be presented as a bilinear combination of dual partial Bethe vectors as follows:*

$$\mathbb{C}_{a,b}(\bar u; \bar v) = \sum r_1^{(1)}(\bar u_{\rm II}) r_3^{(2)}(\bar v_{\rm I}) \frac{f(\bar u_{\rm I}, \bar u_{\rm II}) f(\bar v_{\rm I}, \bar v_{\rm II})}{f(\bar v_{\rm I}, \bar u_{\rm II})} \mathbb{C}^{(1)}_{a_1,b_1}(\bar u_{\rm I}; \bar v_{\rm I}) \mathbb{C}^{(2)}_{a_2,b_2}(\bar u_{\rm II}; \bar v_{\rm II}). \tag{2.18}$$

Here the sums are taken over partitions $\bar u \Rightarrow \{\bar u_{\rm I}, \bar u_{\rm II}\}$ and $\bar v \Rightarrow \{\bar v_{\rm I}, \bar v_{\rm II}\}$. The integers a_ℓ and b_ℓ ($\ell = 1, 2$) are the cardinalities of the corresponding subsets. Hereby, $a_1 + a_2 = a$ and $b_1 + b_2 = b$.

Proof. Using (2.10) we act with the antimorphism ψ on (2.12). Since the action of ψ exchanges the components of the tensor product, we should replace in the r.h.s. the vector $\mathbb{B}^{(1)}$ by $\mathbb{C}^{(2)}$ and the vector $\mathbb{B}^{(2)}$ by $\mathbb{C}^{(1)}$. The subtlety is that we also should make the replacement of the functions $r_k^{(\ell)}$: $r_k^{(1)} \leftrightarrow r_k^{(2)}$. This can be easily understood if we remember the origin of these functions in (2.12). Due to (2.7) total Bethe vectors are polynomials in operators T_{ij} with $i < j$ acting on the vacuum vector

$$\mathbb{B}_{a,b}(\bar u; \bar v) = P(T_{ij})|0\rangle,$$

which implies

$$\mathbb{B}_{a,b}(\bar u; \bar v) = P\left(\sum_{k=1}^{3} T_{ik}^{(2)} T_{kj}^{(1)}\right)|0\rangle.$$

Therefore, in spite of the total Bethe vector depends only on T_{ij} with $i<j$, it depends on the partial $T^{(\ell)}_{ij}$ with $i=j$ as well. It is the action of these operators on the vacuum vector that produces the functions $r^{(\ell)}_i$. But since the action of ψ exchanges the components of the tensor product, we obtain that $T^{(1)}_{ii}\leftrightarrow T^{(2)}_{ii}$, and hence, $r^{(1)}_i\leftrightarrow r^{(2)}_i$. Thus, we obtain

$$\mathbb{C}_{a,b}(\bar u;\bar v)=\sum r^{(1)}_1(\bar u_{\rm I})r^{(2)}_3(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}\mathbb{C}^{(1)}_{a_2,b_2}(\bar u_{\rm II};\bar v_{\rm II})\mathbb{C}^{(2)}_{a_1,b_1}(\bar u_{\rm I};\bar v_{\rm I}).$$

It remains to relabel the subsets $\bar u_{\rm I}\leftrightarrow\bar u_{\rm II}$, $\bar v_{\rm I}\leftrightarrow\bar v_{\rm II}$ and the subscripts of the partial dual Bethe vectors $a_1\leftrightarrow a_2$, $b_1\leftrightarrow b_2$, and we arrive at (2.18). ■

3 Action of $T_{13}(z)$

In order to study the action of the operators $T_{12}(u)$ and $T_{13}(u)$ onto partial Bethe vectors we use (1.1) and present these operators in the form

$$\begin{aligned}T_{12}(u)&=T^{(2)}_{11}(u)T^{(1)}_{12}(u)+T^{(2)}_{12}(u)T^{(1)}_{22}(u)+T^{(2)}_{13}(u)T^{(1)}_{32}(u),\\ T_{13}(u)&=T^{(2)}_{11}(u)T^{(1)}_{13}(u)+T^{(2)}_{12}(u)T^{(1)}_{23}(u)+T^{(2)}_{13}(u)T^{(1)}_{33}(u).\end{aligned}\tag{3.1}$$

Let $\{z,\bar u\}=\bar\eta$ and $\{z,\bar v\}=\bar\xi$. Then due to (2.15) we should prove that

$$\frac{T_{13}(z)}{\lambda_2(z)}\mathcal{B}_{a-1,b-1}(\bar u;\bar v)=\mathcal{B}_{a,b}(\bar\eta;\bar\xi).\tag{3.2}$$

Due to (2.13) a vector $\mathcal{B}_{a,b}(\bar\eta;\bar\xi)$ has the form

$$\mathcal{B}_{a,b}(\bar\eta;\bar\xi)=\sum r^{(2)}_1(\bar\eta_{\rm I})r^{(1)}_3(\bar\xi_{\rm II})\frac{f(\bar\eta_{\rm II},\bar\eta_{\rm I})f(\bar\xi_{\rm II},\bar\xi_{\rm I})}{f(\bar\xi_{\rm II},\bar\eta_{\rm I})}\mathbb{B}^{(1)}(\bar\eta_{\rm I};\bar\xi_{\rm I})\mathbb{B}^{(2)}(\bar\eta_{\rm II};\bar\xi_{\rm II}).\tag{3.3}$$

Remark 3.1. We have omitted the subscripts of the partial Bethe vectors in the r.h.s. of (3.3), because in this case they do not give any additional information. Indeed, the subscripts of partial Bethe vectors are equal to the cardinalities of the subsets of Bethe parameters. Since in (3.3), the sum over partitions is taken over all possible subsets, it is clear that the corresponding cardinalities run through all possible values from 0 to a for the subsets of $\bar u$ and from 0 to b for the subsets of $\bar v$. Therefore below we shall omit the subscripts of the partial Bethe vectors in the sums over partitions.

Consider how the parameter z may enter the subsets of $\bar\eta$ and $\bar\xi$. Obviously, there are three cases in the r.h.s.:

(i) $\bar\eta_{\rm I}=\{z,\bar u_{\rm I}\}$, $\bar\xi_{\rm I}=\{z,\bar v_{\rm I}\}$, $\bar\eta_{\rm II}=\bar u_{\rm II}$, $\bar\xi_{\rm II}=\bar v_{\rm II}$,
(ii) $\bar\eta_{\rm I}=\bar u_{\rm I}$, $\bar\xi_{\rm I}=\bar v_{\rm I}$, $\bar\eta_{\rm II}=\{z,\bar u_{\rm II}\}$, $\bar\xi_{\rm II}=\{z,\bar v_{\rm II}\}$,
(iii) $\bar\eta_{\rm I}=\bar u_{\rm I}$, $\bar\xi_{\rm I}=\{z,\bar v_{\rm I}\}$, $\bar\eta_{\rm II}=\{z,\bar u_{\rm II}\}$, $\bar\xi_{\rm II}=\bar v_{\rm II}$.

The case $z\in\bar\eta_{\rm I}$ and $z\in\bar\xi_{\rm II}$ gives vanishing contribution, because the product $f(\bar\xi_{\rm II},\bar\eta_{\rm I})^{-1}$ contains the factor $f(z,z)^{-1}=0$. Thus, we obtain

$$\mathcal{B}_{a,b}(\bar\eta;\bar\xi)=A_1+A_2+A_3,$$

where

$$A_1=\sum r^{(2)}_1(z)r^{(2)}_1(\bar u_{\rm I})r^{(1)}_3(\bar v_{\rm II})\frac{f(\bar u_{\rm II},z)f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}$$

$$\times \mathbb{B}^{(1)}(\{\bar u_{\rm I}, z\}; \{\bar v_{\rm I}, z\})\mathbb{B}^{(2)}(\bar u_{\rm II}; \bar v_{\rm II}), \tag{3.4}$$

$$A_2 = \sum r_3^{(1)}(z) r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I}) f(z, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})}$$
$$\times \mathbb{B}^{(1)}(\bar u_{\rm I}; \bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II}, z\}; \{\bar v_{\rm II}, z\}), \tag{3.5}$$

$$A_3 = \sum r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(z, \bar u_{\rm I}) f(\bar v_{\rm II}, z) f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})}$$
$$\times \mathbb{B}^{(1)}(\bar u_{\rm I}; \{\bar v_{\rm I}, z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II}, z\}; \bar v_{\rm II}). \tag{3.6}$$

Consider now the action of the operator $T_{13}(z)$ onto the vector $\mathcal{B}_{a-1,b-1}(\bar u;\bar v)$ in the l.h.s. of (3.2). Due to (3.1) we have

$$\frac{T_{13}(z)}{\lambda_2(z)}\mathcal{B}_{a-1,b-1}(\bar u;\bar v) = \left(\frac{T_{11}^{(2)}(z)T_{13}^{(1)}(z)}{\lambda_2^{(2)}(z)\lambda_2^{(1)}(z)} + \frac{T_{12}^{(2)}(z)T_{23}^{(1)}(z)}{\lambda_2^{(2)}(z)\lambda_2^{(1)}(z)} + \frac{T_{13}^{(2)}(z)T_{33}^{(1)}(z)}{\lambda_2^{(2)}(z)\lambda_2^{(1)}(z)}\right)$$
$$\times \mathcal{B}_{a-1,b-1}(\bar u;\bar v).$$

Substituting here (2.13) for $\mathcal{B}_{a-1,b-1}(\bar u;\bar v)$ we find

$$\frac{T_{13}(z)}{\lambda_2(z)}\mathcal{B}_{a-1,b-1}(\bar u;\bar v) = C_1 + C_2 + C_3,$$

where

$$C_k = \sum r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})} \frac{T_{k3}^{(1)}(z)}{\lambda_2^{(1)}(z)} \mathbb{B}^{(1)}(\bar u_{\rm I}; \bar v_{\rm I}) \frac{T_{1k}^{(2)}(z)}{\lambda_2^{(2)}(z)} \mathbb{B}^{(2)}(\bar u_{\rm II}; \bar v_{\rm II}),$$
$$k = 1, 2, 3.$$

Using formulas for the action of the monodromy matrix elements onto Bethe vectors (see Appendix A) we can find the terms C_k explicitly.

Due to (A.1) and (A.4) we have

$$C_1 = \sum_{\substack{\bar u \Rightarrow \{\bar u_{\rm I}, \bar u_{\rm II}\} \\ \bar v \Rightarrow \{\bar v_{\rm I}, \bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})} \mathbb{B}^{(1)}(\{\bar u_{\rm I}, z\}; \{\bar v_{\rm I}, z\})$$
$$\times \Bigg(r_1^{(2)}(z) f(\bar u_{\rm II}, z)\mathbb{B}^{(2)}(\bar u_{\rm II}; \bar v_{\rm II})$$
$$+ f(\bar v_{\rm II}, z) \sum_{\bar u_{\rm II} \Rightarrow \{u_{\rm i}, \bar u_{\rm ii}\}} r_1^{(2)}(u_{\rm i}) g(z, u_{\rm i}) \frac{f(\bar u_{\rm ii}, u_{\rm i})}{f(\bar v_{\rm II}, u_{\rm i})} \mathbb{B}^{(2)}(\{\bar u_{\rm ii}, z\}; \bar v_{\rm II})$$
$$+ \sum_{\substack{\bar u_{\rm II} \Rightarrow \{u_{\rm i}, \bar u_{\rm ii}\} \\ \bar v_{\rm II} \Rightarrow \{v_{\rm i}, \bar v_{\rm ii}\}}} r_1^{(2)}(u_{\rm i}) g(z, v_{\rm i}) g(v_{\rm i}, u_{\rm i}) \frac{f(\bar u_{\rm ii}, u_{\rm i}) f(\bar v_{\rm ii}, v_{\rm i})}{f(\bar v_{\rm II}, u_{\rm i})} \mathbb{B}^{(2)}(\{\bar u_{\rm ii}, z\}; \{\bar v_{\rm ii}, z\})\Bigg). \tag{3.7}$$

Here the original sets of the Bethe parameters are divided into subsets $\bar u \Rightarrow \{\bar u_{\rm I}, \bar u_{\rm II}\}$ and $\bar v \Rightarrow \{\bar v_{\rm I}, \bar v_{\rm II}\}$. Then in some terms the subsets $\bar u_{\rm II}$ and $\bar v_{\rm II}$ are divided into additional subsets $\bar u_{\rm II} \Rightarrow \{u_{\rm i}, \bar u_{\rm ii}\}$ and $\bar v_{\rm II} \Rightarrow \{v_{\rm i}, \bar v_{\rm ii}\}$ with $\#u_{\rm i} = \#v_{\rm i} = 1$ (for this reason we do not write bar for $u_{\rm i}$ and $v_{\rm i}$). The sum is taken over all partitions described above.

Similarly, due to (A.1), (A.2), (A.3), and (A.6) we obtain

$$C_2 = \sum_{\substack{\bar u \Rightarrow \{\bar u_{\rm I}, \bar u_{\rm II}\} \\ \bar v \Rightarrow \{\bar v_{\rm I}, \bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I}) r_3^{(1)}(\bar v_{\rm II}) \frac{f(\bar u_{\rm II}, \bar u_{\rm I}) f(\bar v_{\rm II}, \bar v_{\rm I})}{f(\bar v_{\rm II}, \bar u_{\rm I})}$$

$$\times\left(f(z,\bar u_{\rm I})\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm I},z\})+\sum_{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}} g(u_{\rm i},z)f(u_{\rm i},\bar u_{\rm ii})\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm I},z\})\right)$$
$$\times\left(f(\bar v_{\rm II},z)\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II})+\sum_{\bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}} g(z,v_{\rm i})f(\bar v_{\rm ii},v_{\rm i})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\})\right), \quad (3.8)$$

and

$$C_3=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}\Bigg(r_3^{(1)}(z)f(z,\bar v_{\rm I})\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})$$
$$+f(z,\bar u_{\rm I})\sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}} r_3^{(1)}(v_{\rm i})g(v_{\rm i},z)\frac{f(v_{\rm i},\bar v_{\rm ii})}{f(v_{\rm i},\bar u_{\rm I})}\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm ii},z\})$$
$$+\sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm I}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} r_3^{(1)}(v_{\rm i})g(u_{\rm i},z)g(v_{\rm i},u_{\rm i})\frac{f(u_{\rm i},\bar u_{\rm ii})f(v_{\rm i},\bar v_{\rm ii})}{f(v_{\rm i},\bar u_{\rm I})}\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm ii},z\})\Bigg)$$
$$\times\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}). \quad (3.9)$$

Looking at (3.7) we see that the sum over partitions involving the product of Bethe vectors $\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II})$ coincides with the term A_1 (3.4). Similarly, the sum in (3.8) involving the product $\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II})$ coincides with A_3 (3.6), while the sum in (3.9) involving the product $\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\})$ coincides with A_2 (3.5). Thus, equation (2.15) will be proved if we show that contributions from other terms in (3.7)–(3.9) vanish. To observe these cancellations we combine together the terms where the Bethe vectors $\mathbb{B}^{(1)}$ and $\mathbb{B}^{(2)}$ depend on the parameter z in the same manner.

There are two terms containing the product of the following type:

$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v''\}).$$

Here $\bar u'$, $\bar u''$ and $\bar v'$, $\bar v''$ are arbitrary subsets of the sets $\bar u$ and $\bar v$ respectively. The first term of this type comes from (3.7)

$$C_{12}=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},u_{\rm i},\bar u_{\rm ii}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm ii},\bar u_{\rm I})f(u_{\rm i},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar v_{\rm II},z)f(\bar u_{\rm ii},u_{\rm i})}{f(\bar v_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},u_{\rm i})}$$
$$\times g(z,u_{\rm i})\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\bar v_{\rm II}). \quad (3.10)$$

The second term is contained in (3.8):

$$C_{22}=\sum_{\substack{\bar u\Rightarrow\{u_{\rm i},\bar u_{\rm ii},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},u_{\rm i})f(\bar u_{\rm II},\bar u_{\rm ii})f(u_{\rm i},\bar u_{\rm ii})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar v_{\rm II},z)}{f(\bar v_{\rm II},u_{\rm i})f(\bar v_{\rm II},\bar u_{\rm ii})}$$
$$\times g(u_{\rm i},z)\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}). \quad (3.11)$$

We claim that $C_{12}+C_{22}=0$. For this purpose we relabel the subsets in equation (3.11), so that the vectors in (3.10) and (3.11) to have the same arguments. Observe that such the replacement is nothing but the change of summation variables. Clearly, we should make the following relabeling: first $\bar u_{\rm ii}\to\bar u_{\rm I}$ and then $\bar u_{\rm II}\to\bar u_{\rm ii}$

$$\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II})\xrightarrow{\bar u_{\rm ii}\to\bar u_{\rm I}}\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}),$$

$$\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II})\xrightarrow{\bar u_{\rm II}\to\bar u_{\rm ii}}\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\bar v_{\rm II}).$$

Then we obtain

$$C_{22}=\sum_{\substack{\bar u\Rightarrow\{u_{\rm i},\bar u_{\rm I},\bar u_{\rm ii}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm ii},u_{\rm i})f(\bar u_{\rm ii},\bar u_{\rm I})f(u_{\rm i},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar v_{\rm II},z)}{f(\bar v_{\rm II},u_{\rm i})f(\bar v_{\rm II},\bar u_{\rm I})}$$
$$\times g(u_{\rm i},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\bar v_{\rm II}).$$

We see that $C_{12}+C_{22}=0$ due to the trivial identity

$$g(z,u_{\rm i})+g(u_{\rm i},z)=0. \tag{3.12}$$

The analysis of other contributions can be done in the similar manner. There are two terms containing the sums over partitions involving the products of the Bethe vectors of the form

$$\mathbb{B}^{(1)}(\{\bar u'\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}).$$

It is easy to check that after appropriate relabeling of the subsets they also cancel each other due to identity (3.12).

Finally, all three terms (3.7)–(3.9) contain the sums over partitions involving the products of the Bethe vectors of the form

$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}).$$

Mutual cancellation of these terms is a bit more subtle. These contributions are

$$C_{13}=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},u_{\rm i},\bar u_{\rm ii}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},v_{\rm i},\bar v_{\rm ii}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm ii})r_3^{(1)}(v_{\rm i})\frac{f(u_{\rm i},\bar u_{\rm I})f(\bar u_{\rm ii},\bar u_{\rm I})f(\bar u_{\rm ii},u_{\rm i})}{f(v_{\rm i},\bar u_{\rm I})f(\bar v_{\rm ii},\bar u_{\rm I})f(\bar v_{\rm ii},\bar u_{\rm I})f(v_{\rm i},u_{\rm i})}$$
$$\times f(\bar v_{\rm ii},v_{\rm i})f(v_{\rm i},\bar v_{\rm I})f(\bar v_{\rm ii},\bar v_{\rm I})g(z,v_{\rm i})g(v_{\rm i},u_{\rm i})\mathbb{B}_1(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}_2(\{\bar u_{\rm ii},z\};\{\bar v_{\rm ii},z\}),$$
$$C_{24}=\sum_{\substack{\bar u\Rightarrow\{u_{\rm i},\bar u_{\rm ii},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},v_{\rm i},\bar v_{\rm ii}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm ii})r_3^{(1)}(v_{\rm i})r_3^{(1)}(\bar v_{\rm ii})\frac{f(\bar u_{\rm II},u_{\rm i})f(\bar u_{\rm II},\bar u_{\rm ii})f(u_{\rm i},\bar u_{\rm ii})}{f(\bar v_{\rm ii},\bar u_{\rm ii})f(\bar v_{\rm ii},u_{\rm i})f(v_{\rm i},\bar u_{\rm ii})f(v_{\rm i},u_{\rm i})}$$
$$\times f(\bar v_{\rm ii},\bar v_{\rm I})f(v_{\rm i},\bar v_{\rm I})f(\bar v_{\rm ii},v_{\rm i})g(z,v_{\rm i})g(u_{\rm i},z)\mathbb{B}_1(\{\bar u_{\rm ii},z\};\{\bar v_{\rm I},z\})\mathbb{B}_2(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\}),$$
$$C_{33}=\sum_{\substack{\bar u\Rightarrow\{u_{\rm i},\bar u_{\rm ii},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{v_{\rm i},\bar v_{\rm ii},\bar v_{\rm II}\}}} r_1^{(2)}(u_{\rm i})r_1^{(2)}(\bar u_{\rm ii})r_3^{(1)}(\bar v_{\rm II})r_3^{(1)}(v_{\rm i})\frac{f(\bar u_{\rm II},u_{\rm i})f(\bar u_{\rm II},\bar u_{\rm ii})f(u_{\rm i},\bar u_{\rm ii})}{f(\bar v_{\rm II},u_{\rm i})f(\bar v_{\rm II},\bar u_{\rm ii})f(v_{\rm i},\bar u_{\rm ii})f(v_{\rm i},u_{\rm i})}$$
$$\times f(v_{\rm i},\bar v_{\rm ii})f(\bar v_{\rm II},\bar v_{\rm ii})f(\bar v_{\rm II},v_{\rm i})g(u_{\rm i},z)g(v_{\rm i},u_{\rm i})\mathbb{B}_1(\{\bar u_{\rm ii},z\};\{\bar v_{\rm ii},z\})\mathbb{B}_2(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}).$$

One should again relabel the subsets in such a way that the Bethe vectors in all three terms have the same arguments. For this we relabel in C_{24} $\bar u_{\rm ii}\to\bar u_{\rm I}$ and then $\bar u_{\rm II}\to\bar u_{\rm ii}$. In C_{33} we make the following replacements: $\bar v_{\rm ii}\to\bar v_{\rm I}$, $\bar u_{\rm ii}\to\bar u_{\rm I}$ and then $\bar v_{\rm II}\to\bar v_{\rm ii}$, $\bar u_{\rm II}\to\bar u_{\rm ii}$. After the relabeling described above we obtain that $C_{13}+C_{24}+C_{33}=0$ due to identity

$$g(z,v_{\rm i})g(v_{\rm i},u_{\rm i})+g(u_{\rm i},z)g(v_{\rm i},u_{\rm i})+g(z,v_{\rm i})g(u_{\rm i},z)=0. \tag{3.13}$$

Thus, equation (2.15) is proved.

4 Action of T_{12}

The action of the operator $T_{12}(z)$ on $\mathcal{B}_{a,b}(\bar u;\bar v)$ can be studied in the similar manner, but it is more cumbersome. Therefore in this section we describe only the main steps, while the details are given in Appendix B.

Let again $\bar\eta=\{z,\bar u\}$, $\bar\xi=\{z,\bar v\}$ with cardinalities $\#\bar u=a-1$ and $\#\bar v=b$. Consider a linear combination

$$D=\sum g(z,\xi_0)\frac{f(\bar\xi_0,\xi_0)}{f(z,\xi_0)}\mathcal{B}_{a,b}(\bar\eta;\bar\xi_0), \tag{4.1}$$

where the sum is taken over partitions $\bar\xi\Rightarrow\{\xi_0,\bar\xi_0\}$ with $\#\xi_0=1$. It is easy to see that this linear combination coincides with the r.h.s. of (2.16). Indeed, at $\xi_0=z$ (4.1) yields the first term in (2.16), and if $\xi_0=v_0$, then (4.1) yields the second term. Due to the definition $\mathcal{B}_{a,b}(\bar\eta;\bar\xi_0)$ we have

$$D=\sum r_1^{(2)}(\bar\eta_{\rm I})r_3^{(1)}(\bar\xi_{\rm II})g(z,\xi_0)\frac{f(\bar\eta_{\rm II},\bar\eta_{\rm I})f(\bar\xi_{\rm II},\bar\xi_{\rm I})f(\bar\xi_{\rm I},\xi_0)f(\bar\xi_{\rm II},\xi_0)}{f(\bar\xi_{\rm II},\bar\eta_{\rm I})f(z,\xi_0)}\mathbb{B}^{(1)}(\bar\eta_{\rm I};\bar\xi_{\rm I})\mathbb{B}^{(2)}(\bar\eta_{\rm II};\bar\xi_{\rm II}), \tag{4.2}$$

where the sum is taken over partitions $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ and $\bar\xi\Rightarrow\{\xi_0,\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\xi_0=1$. As before, we do not write the subscripts for the Bethe vectors $\mathbb{B}^{(\ell)}$. Consider again how the parameter z may enter the subsets in (4.2). This time, there are five cases:

(i)	$\bar\eta_{\rm I}=\{z,\bar u_{\rm I}\}$,	$\bar\eta_{\rm II}=\bar u_{\rm II}$,	$\bar\xi_{\rm I}=\bar v_{\rm I}$,	$\bar\xi_{\rm II}=\bar v_{\rm II}$,	$\xi_0=z$,
(ii)	$\bar\eta_{\rm I}=\{z,\bar u_{\rm I}\}$,	$\bar\eta_{\rm II}=\bar u_{\rm II}$,	$\bar\xi_{\rm I}=\{z,\bar v_{\rm I}\}$,	$\bar\xi_{\rm II}=\bar v_{\rm II}$,	$\xi_0=v_0$,
(iii)	$\bar\eta_{\rm I}=\bar u_{\rm I}$,	$\bar\eta_{\rm II}=\{z,\bar u_{\rm II}\}$,	$\bar\xi_{\rm I}=\bar v_{\rm I}$,	$\bar\xi_{\rm II}=\bar v_{\rm II}$,	$\xi_0=z$,
(iv)	$\bar\eta_{\rm I}=\bar u_{\rm I}$,	$\bar\eta_{\rm II}=\{z,\bar u_{\rm II}\}$,	$\bar\xi_{\rm I}=\{z,\bar v_{\rm I}\}$,	$\bar\xi_{\rm II}=\bar v_{\rm II}$,	$\xi_0=v_0$,
(v)	$\bar\eta_{\rm I}=\bar u_{\rm I}$,	$\bar\eta_{\rm II}=\{z,\bar u_{\rm II}\}$,	$\bar\xi_{\rm I}=\bar v_{\rm I}$,	$\bar\xi_{\rm II}=\{z,\bar v_{\rm II}\}$,	$\xi_0=v_0$.

In all these five cases we have different contributions to the linear combination D. So, we have

$$D=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}}G_{\rm I,II}(D_1+D_3)+\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{v_0,\bar v_{\rm I},\bar v_{\rm II}\}}}G_{0,\rm I,II}(D_2+D_4+D_5), \tag{4.3}$$

where

$$G_{\rm I,II}=r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})},$$

$$G_{0,\rm I,II}=r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})g(z,v_0)f(\bar v_{\rm II},v_0)f(\bar v_{\rm I},v_0)\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})},$$

and

$$D_1=r_1^{(2)}(z)f(\bar u_{\rm II},z)f(\bar v_{\rm I},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}), \tag{4.4}$$

$$D_2=r_1^{(2)}(z)f(\bar u_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}), \tag{4.5}$$

$$D_3=(z,\bar u_{\rm I})f(\bar v,z)\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{4.6}$$

$$D_4=f(z,\bar u_{\rm I})f(\bar v_{\rm II},z)\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{4.7}$$

$$D_5=r_3^{(1)}(z)f(z,\bar v_{\rm I})\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}). \tag{4.8}$$

In all these formulas the set $\bar u$ is divided into subsets $\{\bar u_{\rm I},\bar u_{\rm II}\}$. As for the set $\bar v$, it is divided either into subsets $\{\bar v_{\rm I},\bar v_{\rm II}\}$ (like in (4.4), (4.6)), or into subsets $\{v_0,\bar v_{\rm I},\bar v_{\rm II}\}$ with $\#v_0=1$ (like in (4.5), (4.7), (4.8)).

We should compare these contributions with the l.h.s. of (2.16). There we have

$$\frac{T_{12}(z)}{\lambda_2(z)}\mathcal{B}_{a-1,b}(\bar u;\bar v) = \left(\frac{T^{(2)}_{11}(z)T^{(1)}_{12}(z)}{\lambda^{(2)}_2(z)\lambda^{(1)}_2(z)} + \frac{T^{(2)}_{12}(z)T^{(1)}_{22}(z)}{\lambda^{(2)}_2(z)\lambda^{(1)}_2(z)} + \frac{T^{(2)}_{13}(z)T^{(1)}_{32}(z)}{\lambda^{(2)}_2(z)\lambda^{(1)}_2(z)}\right)\mathcal{B}_{a-1,b}(\bar u;\bar v).$$

Thus, we obtain

$$\frac{T_{12}(z)}{\lambda_2(z)}\mathcal{B}_{a-1,b}(\bar u;\bar v) = E_1 + E_2 + E_3, \tag{4.9}$$

where for $k = 1, 2, 3$

$$E_k = \sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r^{(2)}_1(\bar u_{\rm I})r^{(1)}_3(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}\frac{T^{(1)}_{k2}(z)}{\lambda^{(1)}_2(z)}\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\frac{T^{(2)}_{1k}(z)}{\lambda^{(2)}_2(z)}\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}).$$

Using again the formulas of Appendix A we obtain explicit expressions for the terms E_k. They are given in Appendix B. Then similarly to the case considered in the previous section we should compare coefficients of the products of the partial Bethe vectors of different type (depending on how the parameter z enters the arguments of Bethe vectors). One can show that five terms in the combination $E_1 + E_2 + E_3$ reproduce the sum (4.3), while other contributions mutually cancel each other. In this way we arrive at equation (2.16).

As we have explained in Section 2.3 the actions (2.15) and (2.16) imply the recursion (2.17). Since this recursion coincides with the one for the Bethe vectors [1], we conclude that $\mathcal{B}_{a,b}(\bar u;\bar v)$ (2.13) is the Bethe vector $\mathbb{B}_{a,b}(\bar u;\bar v)$.

5 Coproduct property for Bethe vectors in GL(N)-based models

In this section we use the results obtained in [2, 6] adapted to the case under consideration.

Let Π be a set $\{1,\dots,N-1\}$ of indices of simple positive roots of $\mathfrak{gl}_N$. Let $I = \{i_1,\dots,i_n\}$ be a finite collection of positive integers. We attach to each multiset I an ordered set of variables $\bar t_I = \{t_i \,|\, i\in I\} = \{t_{i_1},\dots,t_{i_n}\}$ with a 'coloring' map $\iota\colon \bar t_I \to \Pi$. For any multiset I we assume a natural linear ordering in the set of variables $\bar t_I$

$$t_{i_1} \prec \cdots \prec t_{i_n}.$$

Each variable t_{i_k} has its own 'type': $\iota(t_{i_k}) \in \Pi$. We call such set of variables $\bar t_I$ an *ordered* Π-*multiset*.

For example, for the GL(3)-based models, the multiset

$$I = \{1,\dots,a,1,\dots,b\} \tag{5.1}$$

is used to parameterize the sets $\bar u$ and $\bar v$ of the Bethe parameters for the Bethe vectors $\mathbb{B}_{a,b}(\bar u;\bar v)$

$$\bar u = \{u_1,\dots,u_a\}, \qquad \bar v = \{v_1,\dots,v_b\}$$

and the corresponding ordering in the set $\bar t_I = \{\bar u,\bar v\}$ is

$$u_1 \prec \cdots \prec u_a \prec v_1 \prec \cdots \prec v_b. \tag{5.2}$$

Given elements t_i, t_j of some ordered Π-multiset define two functions $\gamma(t_i,t_j)$ and $\beta(t_i,t_j)$ by the formulas

$$\gamma(t_i,t_j)=\begin{cases} f(t_i,t_j)^{-1}, & \text{if} \quad \iota(t_i)=\iota(t_j)+1,\\ f(t_j,t_i), & \text{if} \quad \iota(t_j)=\iota(t_i)+1,\\ 1, & \text{otherwise,}\end{cases} \qquad \beta(t_i,t_j)=\begin{cases} f(t_j,t_i), & \text{if} \quad \iota(t_i)=\iota(t_j),\\ 1, & \text{otherwise.}\end{cases}$$

Let $V=V^{(1)}\otimes V^{(2)}$ be a tensor product of two representations with vacuum vectors $|0\rangle^{(1)}$, $|0\rangle^{(2)}$ and weight series $\{\lambda_b^{(1)}(u)\}$ and $\{\lambda_b^{(2)}(u)\}$, $b=1,\dots,N$.

A collection of rational V-valued functions $\mathbf{w}_{V,I}(t_i|_{i\in I})$, depending on the representation V with a weight vacuum vector $|0\rangle$, and an ordered Π-multiset $\bar t_I$, is called a *modified weight function* $\mathbf{w}$ (it is the same as Bethe vectors up to normalization), if the following relations are satisfied

- $\mathbf{w}_{V,\varnothing}\equiv|0\rangle=|0\rangle_1\otimes|0\rangle_2$
- and

$$\mathbf{w}_{V,I}(t_i|_{i\in I})=\sum_{I\Rightarrow\{I_1,I_2\}}\mathbf{w}_{V_1,I_1}(t_i|_{i\in I_1})\otimes\mathbf{w}_{V_2,I_2}(t_i|_{i\in I_2})\cdot\Phi_{I_1,I_2}(t_i|_{i\in I})$$
$$\times\prod_{j\in I_1}\lambda^{(2)}_{\iota(t_j)}(t_j)\prod_{j\in I_2}\lambda^{(1)}_{\iota(t_j)+1}(t_j), \tag{5.3}$$

where

$$\Phi_{I_1,I_2}(t_i|_{i\in I})=\prod_{i\in I_1,\ j\in I_2}\beta(t_i,t_j)\prod_{i\in I_2,\ j\in I_1,\ t_i\prec t_j}\gamma(t_i,t_j).$$

The summation in (5.3) runs over all possible decompositions of the multiset I into a disjoint union of two non-intersecting multisubsets $I\Rightarrow\{I_1,I_2\}$. The structure of ordered Π-multiset on each multisubset is induced from that of I.

For example, in the GL(3)-based model the decomposition of the multiset I (5.1) into two non-intersecting multisubsets I_1 and I_2 corresponds to the decompositions of the sets $\bar u$ and $\bar v$ into two pairs of the subsets

$$\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\} \qquad \text{and} \qquad \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\} \tag{5.4}$$

with the mutual ordering

$$\bar u_{\rm I}\prec\bar v_{\rm I},\qquad \bar u_{\rm I}\prec\bar v_{\rm II},\qquad \bar u_{\rm II}\prec\bar v_{\rm I},\qquad \bar u_{\rm II}\prec\bar v_{\rm II}.$$

inherited by the ordering (5.2). So the multiset I_1 is such that $\bar t_{I_1}=\{\bar u_{\rm I},\bar v_{\rm I}\}$ and the corresponding to the multiset I_2 is a subset of variables $\bar t_{I_2}=\{\bar u_{\rm II},\bar v_{\rm II}\}$.

The function $\Phi_{I_1,I_2}(t_i|_{i\in I})$ in this case is

$$\Phi(\bar u_{\rm I},\bar u_{\rm II},\bar v_{\rm I},\bar v_{\rm II})=f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar v_{\rm I},\bar u_{\rm II}),$$

and the formula (5.3) becomes

$$\mathbf{w}_V(\bar u;\bar v)=\sum\mathbf{w}_{V_1}(\bar u_{\rm I};\bar v_{\rm I})\otimes\mathbf{w}_{V_2}(\bar u_{\rm II};\bar v_{\rm II})$$
$$\times f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})f(\bar v_{\rm I},\bar u_{\rm II})\lambda_1^{(2)}(\bar u_{\rm I})\lambda_2^{(2)}(\bar v_{\rm I})\lambda_2^{(1)}(\bar u_{\rm II})\lambda_3^{(1)}(\bar u_{\rm II}), \tag{5.5}$$

where the sum is taken over the partitions (5.4).

Define now the normalization of the Bethe vectors that we used for the GL(3)-based models in [1]

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \frac{\mathbf{w}_V(\bar{u};\bar{v})}{f(\bar{v},\bar{u})\lambda_2(\bar{u})\lambda_2(\bar{v})}. \tag{5.6}$$

Then, taking into account the coproduct property of the diagonal monodromy matrix elements (2.3)

$$\lambda_2(\bar{u}) = \lambda_2^{(1)}(\bar{u})\lambda_2^{(2)}(\bar{u}) = \lambda_2^{(1)}(\bar{u}_{\rm I})\lambda_2^{(1)}(\bar{u}_{\rm II})\lambda_2^{(2)}(\bar{u}_{\rm I})\lambda_2^{(2)}(\bar{u}_{\rm II}),$$
$$\lambda_2(\bar{v}) = \lambda_2^{(1)}(\bar{v})\lambda_2^{(2)}(\bar{v}) = \lambda_2^{(1)}(\bar{v}_{\rm I})\lambda_2^{(1)}(\bar{v}_{\rm II})\lambda_2^{(2)}(\bar{v}_{\rm I})\lambda_2^{(2)}(\bar{v}_{\rm II}),$$

we recast (5.5) as

$$\frac{\mathbf{w}_V(\bar{u};\bar{v})}{f(\bar{v},\bar{u})\lambda_2(\bar{u})\lambda_2(\bar{v})} = \sum \frac{\mathbf{w}_{V_1}(\bar{u}_{\rm I};\bar{v}_{\rm I})}{f(\bar{v}_{\rm I},\bar{u}_{\rm I})\lambda_2^{(1)}(\bar{u}_{\rm I})\lambda_2^{(1)}(\bar{v}_{\rm I})} \otimes \frac{\mathbf{w}_{V_2}(\bar{u}_{\rm II};\bar{v}_{\rm II})}{f(\bar{v}_{\rm II},\bar{u}_{\rm II})\lambda_2^{(2)}(\bar{u}_{\rm II})\lambda_2^{(2)}(\bar{v}_{\rm II})}$$
$$\times \frac{f(\bar{u}_{\rm II},\bar{u}_{\rm I})f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{f(\bar{v}_{\rm II},\bar{u}_{\rm I})} \cdot r_1^{(2)}(\bar{u}_{\rm I})r_3^{(1)}(\bar{v}_{\rm II}).$$

Under identification (5.6) this relation coincides with the statement of the Theorem 2.1.

6 Conclusion

A composite model allows one to apply the algebraic Bethe ansatz to the study of operators depending on an internal point of the original interval. In particular, one can compute form factors of these operators and then use the results obtained for calculating correlation functions. A formula for the Bethe vector is a necessary tool of this program.

In this paper we obtained an explicit expression for the Bethe vectors in the composite quantum integrable model associated with GL(3)-invariant R-matrix (2.1). We did it in two ways. The first one is based on the action of the monodromy matrix elements onto Bethe vectors obtained in [1]. The second way is based on the coproduct property of the weight functions proved by the completely different methods in [16]. Thus, we have shown that the formula that relates Bethe vector of the composite model with the partial Bethe vectors is equivalent to the coproduct property.

The next step of the program is to use the formulas for the Bethe vectors in the composite model for calculating form factors of the entries of the partial monodromy matrices $T_{ij}^{(\ell)}$. This will be the subject of our further publication [13]. There we consider form factors of zero modes [12] of the operators $T_{ij}^{(\ell)}$. We show that they can be reduced to the form factors of total monodromy matrix entries T_{ij}. In this way we obtain determinant representations for form factors of local operators in GL(3)-invariant models.

A Formulas of the monodromy matrix elements action onto Bethe vectors

We give here a list of formulas of the action of $T_{ij}(z)$ on Bethe vectors. These formulas were obtained in the paper [1] for the more general situation of multiple action of the monodromy matrix elements onto Bethe vectors. In [1] these formulas were proved by induction and using the explicit formulas for the Bethe vectors (2.7).

Formulas for the actions of the partial $T_{ij}^{(\ell)}(z)$ on partial Bethe vectors $\mathbb{B}_{a,b}^{(\ell)}$ are the same up to the replacements of λ_2 and r_k by $\lambda_2^{(\ell)}$ and $r_k^{(\ell)}$ respectively. All formulas (except (A.1)) are

given in terms of sums over partitions of the original sets of the Bethe parameters into subsets. In all the formulas $\#u_0 = \#v_0 = 1$, and we recall that $\bar u_0 = \bar u \setminus u_0$ and $\bar v_0 = \bar v \setminus v_0$.

- Action of $T_{13}(z)$:

$$\frac{T_{13}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = \mathbb{B}_{a+1,b+1}(\{\bar u, z\};\{\bar v, z\}). \tag{A.1}$$

- Action of $T_{12}(z)$:

$$\begin{aligned}\frac{T_{12}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = {}& f(\bar v, z)\mathbb{B}_{a+1,b}(\{\bar u, z\};\bar v)\\ &+\sum_{\bar v\Rightarrow\{v_0,\bar v_0\}} g(z,v_0)f(\bar v_0,v_0)\mathbb{B}_{a+1,b}(\{\bar u, z\};\{\bar v_0, z\}).\end{aligned} \tag{A.2}$$

- Action of $T_{23}(z)$:

$$\begin{aligned}\frac{T_{23}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = {}& f(z,\bar u)\mathbb{B}_{a,b+1}(\bar u;\{\bar v, z\})\\ &+\sum_{\bar u\Rightarrow\{u_0,\bar u_0\}} g(u_0,z)f(u_0,\bar u_0)\mathbb{B}_{a,b+1}(\{\bar u_0, z\};\{\bar v, z\}).\end{aligned} \tag{A.3}$$

- Action of $T_{11}(z)$:

$$\begin{aligned}\frac{T_{11}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = {}& r_1(z)f(\bar u,z)\mathbb{B}_{a,b}(\bar u;\bar v)\\ &+ f(\bar v,z)\sum_{\bar u\Rightarrow\{u_0,\bar u_0\}} r_1(u_0)g(z,u_0)\frac{f(\bar u_0,u_0)}{f(\bar v,u_0)}\mathbb{B}_{a,b}(\{\bar u_0, z\};\bar v)\\ &+\sum_{\substack{\bar u\Rightarrow\{u_0,\bar u_0\}\\ \bar v\Rightarrow\{v_0,\bar v_0\}}} r_1(u_0)g(z,v_0)g(v_0,u_0)\frac{f(\bar u_0,u_0)f(\bar v_0,v_0)}{f(\bar v,u_0)}\mathbb{B}_{a,b}(\{\bar u_0, z\};\{\bar v_0, z\}).\end{aligned} \tag{A.4}$$

- Action of $T_{22}(z)$:

$$\begin{aligned}\frac{T_{22}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = {}& f(\bar v,z)f(z,\bar u)\mathbb{B}_{a,b}(\bar u;\bar v)\\ &+ f(z,\bar u)\sum_{\bar v\Rightarrow\{v_0,\bar v_0\}} g(z,v_0)f(\bar v_0,v_0)\mathbb{B}_{a,b}(\bar u;\{\bar v_0, z\})\\ &+ f(\bar v,z)\sum_{\bar u\Rightarrow\{u_0,\bar u_0\}} g(u_0,z)f(u_0,\bar u_0)\mathbb{B}_{a,b}(\{\bar u_0, z\};\bar v)\\ &+\sum_{\substack{\bar u\Rightarrow\{u_0,\bar u_0\}\\ \bar v\Rightarrow\{v_0,\bar v_0\}}} g(z,v_0)g(u_0,z)f(u_0,\bar u_0)f(\bar v_0,v_0)\mathbb{B}_{a,b}(\{\bar u_0, z\};\{\bar v_0, z\}).\end{aligned} \tag{A.5}$$

- Action of $T_{33}(z)$:

$$\begin{aligned}\frac{T_{33}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v) = {}& r_3(z)f(z,\bar v)\mathbb{B}_{a,b}(\bar u;\bar v)\\ &+ f(z,\bar u)\sum_{\bar v\Rightarrow\{v_0,\bar v_0\}} r_3(v_0)g(v_0,z)\frac{f(v_0,\bar v_0)}{f(v_0,\bar u)}\mathbb{B}_{a,b}(\bar u;\{\bar v_0, z\})\end{aligned}$$

$$+\sum_{\substack{\bar u\Rightarrow\{u_0,\bar u_0\}\\ \bar v\Rightarrow\{v_0,\bar v_0\}}} r_3(v_0)g(u_0,z)g(v_0,u_0)\frac{f(u_0,\bar u_0)f(v_0,\bar v_0)}{f(v_0,\bar u)}\mathbb{B}_{a,b}(\{\bar u_0,z\};\{\bar v_0,z\}). \tag{A.6}$$

- Action of $T_{32}(z)$:

$$\begin{aligned}
&\frac{T_{32}(z)}{\lambda_2(z)}\mathbb{B}_{a,b}(\bar u;\bar v)\\
&\quad=\sum_{\substack{\bar u\Rightarrow\{u_0,\bar u_0\}\\ \bar v\Rightarrow\{v_0,\bar v_0\}}} r_3(v_0)g(u_0,z)g(v_0,u_0)\frac{f(u_0,\bar u_0)f(v_0,\bar v_0)f(\bar v_0,z)}{f(v_0,\bar u)}\mathbb{B}_{a,b-1}(\{\bar u_0,z\};\bar v_0)\\
&\qquad+\sum_{\bar v\Rightarrow\{v_0,\bar v_0\}} g(z,v_0)\Big(r_3(z)f(z,\bar v_0)f(\bar v_0,v_0)-r_3(v_0)\frac{f(\bar v_0,z)f(v_0,\bar v_0)f(z,\bar u)}{f(v_0,\bar u)}\Big)\\
&\qquad\quad\times\mathbb{B}_{a,b-1}(\bar u;\bar v_0)\\
&\quad+\sum_{\bar v\Rightarrow\{v_0,v_1,\bar v_2\}} r_3(v_0)g(v_0,z)g(z,v_1)\frac{f(v_0,v_1)f(v_0,\bar v_2)f(\bar v_2,v_1)f(z,\bar u)}{f(v_0,\bar u)}\\
&\qquad\quad\times\mathbb{B}_{a,b-1}(\bar u;\{\bar v_2,z\})\\
&\qquad+\sum_{\substack{\bar u\Rightarrow\{u_0,\bar u_0\}\\ \bar v\Rightarrow\{v_0,v_1\bar v_2\}}} r_3(v_0)g(u_0,z)g(z,v_1)g(v_0,u_0)\frac{f(v_0,v_1)f(v_0,\bar v_2)f(\bar v_2,v_1)f(u_0,\bar u_0)}{f(v_0,\bar u)}\\
&\qquad\quad\times\mathbb{B}_{a,b-1}(\{\bar u_0,z\};\{\bar v_2,z\}).
\end{aligned}\tag{A.7}$$

In this formula $\#v_0=\#v_1=\#u_0=1$.

B Proof of the equality (2.16)

In this appendix we give explicit formulas for the terms E_k in (4.9). It is clear from the formulas of Appendix A that E_1, E_2 and E_3 respectively consist of six, eight and five different sums over partitions. We denote them as $E_{1,m}$, $m=1,\dots,6$, $E_{2,m}$, $m=1,\dots,8$, and $E_{3,m}$, $m=1,\dots,5$. Each of these contributions can be presented in the form

$$E_{k,m}=\sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}\gamma_{k,m}, \tag{B.1}$$

where $\gamma_{k,m}$ may contain additional sums over partitions.

Due to (A.2) and (A.4) we have for the term E_1:

$$\gamma_{1,1}=r_1^{(2)}(z)f(\bar v_{\rm I},z)f(\bar u_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}), \tag{B.2}$$

$$\begin{aligned}\gamma_{1,2}=&\sum_{\bar u_{\rm II}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}} r_1^{(2)}(u_{\rm i})g(z,u_{\rm i})\frac{f(\bar v_{\rm I},z)f(\bar v_{\rm II},z)f(\bar u_{\rm ii},u_{\rm i})}{f(\bar v_{\rm II},u_{\rm i})}\\
&\times\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\bar v_{\rm II}),\end{aligned} \tag{B.3}$$

$$\begin{aligned}\gamma_{1,3}=&\sum_{\substack{\bar u_{\rm II}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} r_1^{(2)}(u_{\rm i})g(z,v_{\rm i})g(v_{\rm i},u_{\rm i})\frac{f(\bar v_{\rm I},z)f(\bar u_{\rm ii},u_{\rm i})f(\bar v_{\rm ii},v_{\rm i})}{f(\bar v_{\rm II},u_{\rm i})}\\
&\times\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm ii},z\}),\end{aligned} \tag{B.4}$$

$$\gamma_{1,4} = \sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}} r_1^{(2)}(z)g(z,v_{\rm iii})f(\bar v_{\rm iv},v_{\rm iii})f(\bar u_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}), \tag{B.5}$$

$$\gamma_{1,5} = \sum_{\substack{\bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}\\ \bar u_{\rm II}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}}} r_1^{(2)}(u_{\rm i})g(z,u_{\rm i})g(z,v_{\rm iii})\frac{f(\bar v_{\rm iv},v_{\rm iii})f(\bar v_{\rm II},z)f(\bar u_{\rm ii},u_{\rm i})}{f(\bar v_{\rm II},u_{\rm i})}$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\bar v_{\rm II}), \tag{B.6}$$

$$\gamma_{1,6} = \sum_{\substack{\bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}\\ \bar u_{\rm II}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} r_1^{(2)}(u_{\rm i})g(z,v_{\rm iii})g(z,v_{\rm i})g(v_{\rm i},u_{\rm i})\frac{f(\bar v_{\rm iv},v_{\rm iii})f(\bar u_{\rm ii},u_{\rm i})f(\bar v_{\rm ii},v_{\rm i})}{f(\bar v_{\rm II},u_{\rm i})}$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm ii},z\}). \tag{B.7}$$

Due to (A.5) and (A.2) we have for the term E_2:

$$\gamma_{2,1} = f(\bar v_{\rm I},z)f(z,\bar u_{\rm I})f(\bar v_{\rm II},z)\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{B.8}$$

$$\gamma_{2,2} = \sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}} g(z,v_{\rm iii})f(z,\bar u_{\rm I})f(\bar v_{\rm iv},v_{\rm iii})f(\bar v_{\rm II},z)\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{B.9}$$

$$\gamma_{2,3} = \sum_{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}} g(u_{\rm i},z)f(\bar v_{\rm I},z)f(u_{\rm i},\bar u_{\rm ii})f(\bar v_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{B.10}$$

$$\gamma_{2,4} = \sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}}} g(z,v_{\rm iii})g(u_{\rm i},z)f(u_{\rm i},\bar u_{\rm ii})f(\bar v_{\rm iv},v_{\rm iii})f(\bar v_{\rm II},z)$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\bar v_{\rm II}), \tag{B.11}$$

$$\gamma_{2,5} = \sum_{\bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}} g(z,v_{\rm i})f(\bar v_{\rm I},z)f(z,\bar u_{\rm I})f(\bar v_{\rm ii},v_{\rm i})\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\}), \tag{B.12}$$

$$\gamma_{2,6} = \sum_{\substack{\bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}\\ \bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} g(z,v_{\rm iii})g(z,v_{\rm i})f(z,\bar u_{\rm I})f(\bar v_{\rm iv},v_{\rm iii})f(\bar v_{\rm ii},v_{\rm i})$$
$$\times\,\mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\}), \tag{B.13}$$

$$\gamma_{2,7} = \sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} g(u_{\rm i},z)g(z,v_{\rm i})f(u_{\rm i},\bar u_{\rm ii})f(\bar v_{\rm I},z)f(\bar v_{\rm ii},v_{\rm i})$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\bar v_{\rm I})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\}), \tag{B.14}$$

$$\gamma_{2,8} = \sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm I}\Rightarrow\{v_{\rm iii},\bar v_{\rm iv}\}\\ \bar v_{\rm II}\Rightarrow\{v_{\rm i},\bar v_{\rm ii}\}}} g(z,v_{\rm iii})g(u_{\rm i},z)g(z,v_{\rm i})f(u_{\rm i},\bar u_{\rm ii})f(\bar v_{\rm iv},v_{\rm iii})f(\bar v_{\rm ii},v_{\rm i})$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm ii},z\}). \tag{B.15}$$

Due to (A.7) and (A.1) we have for the term E_3:

$$\gamma_{3,1} = \sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm I}\Rightarrow\{v_{\rm i},\bar v_{\rm iii}\}}} r_3^{(1)}(v_{\rm i})g(u_{\rm i},z)g(v_{\rm i},u_{\rm i})\frac{f(u_{\rm i},\bar u_{\rm ii})f(v_{\rm i},\bar v_{\rm iii})f(\bar v_{\rm iii},z)}{f(v_{\rm i},\bar u_{\rm I})}$$
$$\times\,\mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\bar v_{\rm iii})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}), \tag{B.16}$$

$$\gamma_{3,2} = \sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm i},\bar v_{\rm iii}\}} r_3^{(1)}(z)g(z,v_{\rm i})f(z,\bar v_{\rm iii})f(\bar v_{\rm iii},v_{\rm i})\mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm iii})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}), \tag{B.17}$$

$$\gamma_{3,3} = \sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm i},\bar v_{\rm iii}\}} r_3^{(1)}(v_{\rm i})g(v_{\rm i},z)\frac{f(\bar v_{\rm iii},z)f(v_{\rm i},\bar v_{\rm iii})f(z,\bar u_{\rm I})}{f(v_{\rm i},\bar u_{\rm I})}$$
$$\times\, \mathbb{B}^{(1)}(\bar u_{\rm I};\bar v_{\rm iii})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}), \tag{B.18}$$
$$\gamma_{3,4} = \sum_{\bar v_{\rm I}\Rightarrow\{v_{\rm i},v_{\rm ii},\bar v_{\rm iii}\}} r_3^{(1)}(v_{\rm i})g(v_{\rm i},z)g(z,v_{\rm ii})\frac{f(v_{\rm i},v_{\rm ii})f(v_{\rm i},\bar v_{\rm iii})f(\bar v_{\rm iii},v_{\rm ii})f(z,\bar u_{\rm I})}{f(v_{\rm i},\bar u_{\rm I})}$$
$$\times\, \mathbb{B}^{(1)}(\bar u_{\rm I};\{\bar v_{\rm iii},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}), \tag{B.19}$$
$$\gamma_{3,5} = \sum_{\substack{\bar u_{\rm I}\Rightarrow\{u_{\rm i},\bar u_{\rm ii}\}\\ \bar v_{\rm I}\Rightarrow\{v_{\rm i},v_{\rm ii}\bar v_{\rm iii}\}}} r_3^{(1)}(v_{\rm i})g(u_{\rm i},z)g(z,v_{\rm ii})g(v_{\rm i},u_{\rm i})\frac{f(v_{\rm i},v_{\rm ii})f(v_{\rm i},\bar v_{\rm iii})f(\bar v_{\rm iii},v_{\rm ii})f(u_{\rm i},\bar u_{\rm ii})}{f(v_{\rm i},\bar u_{\rm I})}$$
$$\times\, \mathbb{B}^{(1)}(\{\bar u_{\rm ii},z\};\{\bar v_{\rm iii},z\})\mathbb{B}^{(2)}(\{\bar u_{\rm II},z\};\{\bar v_{\rm II},z\}). \tag{B.20}$$

First of all one can easily observe that contributions $\gamma_{1,1}$ (B.2) and $\gamma_{2,1}$ (B.8) into the sum (4.9) coincide identically with the contributions D_1 (4.4) and D_3 (4.6) into the first sum of (4.3). The contributions $\gamma_{1,4}$ (B.5), $\gamma_{2,2}$ (B.9) and $\gamma_{3,2}$ (B.17) coincide with the contributions D_2 (4.5), D_4 (4.7) and D_5 (4.8) into the second sum of (4.3) up to a relabeling of the subsets. Indeed, consider, for example, the contribution $E_{1,4}$. Combining (B.1) and (B.5) we obtain

$$E_{1,4} = \sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{v_{\rm iii},\bar v_{\rm iv},\bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},v_{\rm iii})f(\bar v_{\rm II},\bar v_{\rm iv})}{f(\bar v_{\rm II},\bar u_{\rm I})}$$
$$\times\, r_1^{(2)}(z)g(z,v_{\rm iii})f(\bar v_{\rm iv},v_{\rm iii})f(\bar u_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm iv},z\})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}).$$

Relabeling of the subsets $v_{\rm iii}\to v_0$ and $\bar v_{\rm iv}\to\bar v_{\rm I}$ leads to

$$E_{1,4} = \sum_{\substack{\bar u\Rightarrow\{\bar u_{\rm I},\bar u_{\rm II}\}\\ \bar v\Rightarrow\{v_0,\bar v_{\rm I},\bar v_{\rm II}\}}} r_1^{(2)}(\bar u_{\rm I})r_3^{(1)}(\bar v_{\rm II})\frac{f(\bar u_{\rm II},\bar u_{\rm I})f(\bar v_{\rm II},v_0)f(\bar v_{\rm II},\bar v_{\rm I})}{f(\bar v_{\rm II},\bar u_{\rm I})}$$
$$\times\, r_1^{(2)}(z)g(z,v_0)f(\bar v_{\rm I},v_0)f(\bar u_{\rm II},z)\mathbb{B}^{(1)}(\{\bar u_{\rm I},z\};\{\bar v_{\rm I},z\})\mathbb{B}^{(2)}(\bar u_{\rm II};\bar v_{\rm II}),$$

which exactly coincides with the sum over partitions (4.3) of the contribution D_2. The identifications of the contributions $E_{2,2}$ and $E_{3,2}$ with the terms D_4 and D_5 in (4.3) can be done similarly.

All other contributions should vanish. To check this we follow the same strategy as in Section 3. We consider separately the coefficients of the products of the Bethe vectors $\mathbb{B}^{(1)}\mathbb{B}^{(2)}$ with a different placement of the parameter z. There are six different types of these products:

$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v'\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v''\}),$$
$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v'\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}),$$
$$\mathbb{B}^{(1)}(\{\bar u'\};\{\bar v'\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}),$$
$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v''\}),$$
$$\mathbb{B}^{(1)}(\{\bar u'\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}),$$
$$\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\}).$$

- Coefficients of $\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v'\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v''\})$ come from (B.3) and (B.10).
- Coefficients of $\mathbb{B}^{(1)}(\{\bar u'\};\{\bar v'\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v'',z\})$ come from (B.12) and (B.18).
- Coefficients of $\mathbb{B}^{(1)}(\{\bar u',z\};\{\bar v',z\})\mathbb{B}^{(2)}(\{\bar u'',z\};\{\bar v''\})$ come from (B.6) and (B.11).

- Coefficients of $\mathbb{B}^{(1)}(\{\bar{u}'\};\{\bar{v}',z\})\mathbb{B}^{(2)}(\{\bar{u}'',z\};\{\bar{v}'',z\})$ come from (B.13) and (B.19).

In all these four cases the cancellations of the coefficients after relabeling the subsets occurs due to the identity (3.12).

Coefficients of $\mathbb{B}^{(1)}(\{\bar{u}',z\};\{\bar{v}'\})\mathbb{B}^{(2)}(\{\bar{u}'',z\};\{\bar{v}'',z\})$ come from (B.4), (B.14) and (B.16). After proper relabeling the subsets they cancel each other due to the identity (3.13).

Finally, the coefficients of $\mathbb{B}^{(1)}(\{\bar{u}',z\};\{\bar{v}',z\})\mathbb{B}^{(2)}(\{\bar{u}'',z\};\{\bar{v}'',z\})$ come from (B.7), (B.15) and (B.20). In this case again after relabeling the subsets they vanish due to the identity (3.13).

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